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HEAT CONDUCTION AND THERMOELASTIC STRESSES

NONIDEAL CONTACT PROBLEM OF NONSTATIONARY HEAT CONDUCTION FOR TWO HALF-SPACES

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A nonideal contact problem of nonstationary heat conduction for two half-spaces with arbitrary initial conditions is considered. By introducing unknown conjugate functions and subsequent Laplace transformation, an integral representation for the problem solution has been found; equations in the region of transforms are solved with the aid of Green's functions. In the particular case of exponential distributions of initial temperatures an accurate solution of the problem has been achieved.

Introduction. At the present time a fair amount of attention has been devoted to the problems of obtaining accurate solutions of heat conduction equations under different boundary-value conditions [1]. Analytical solutions are of interest, first, purely scientifically and, second, as test problems in calculations by numerical schemes and for the development of the latter. In the theory of heat conduction, nonideally contact boundary-value problems have been studied to a considerably lesser extent. The development of numerical schemes for them is at the initial stage now, so that any analytical investigations in the field are of interest. Note that in the case of an ideal contact of half-spaces, where their initial temperatures are given by arbitrary smooth functions of the coordinate, it seems that the boundary-value problem of heat conduction has been solved for the first time by Datsev [2]. Thereafter it was also tackled by other researchers by other methods, e.g., [3, 4]. In what follows, an analogous statement of the problem is considered in application to a nonideal contact of half-spaces; an integral representation of temperature distributions has been obtained. The method of Laplace transformations in the time domain is used for the solution. The boundary-value problem in the region of transforms is solved by the method of Green's functions. In the particular case, the equations obtained yield accurate solutions of the problem with exponential and constant initial distributions of temperatures.

Statement and Solution of the Problem. Let a contact surface be located at the coordinate origin $x = 0$. We assume that the temperature of the half-spaces at the initial instant of time is given in the form of smooth functions $T_{10}(x)$ and $T_{20}(x)$. Then the problem is described by the heat conduction equations

$$\frac{\partial T_1}{\partial t} = a_1 \frac{\partial^2 T_1}{\partial x^2}\,, \quad x < 0\,, \quad t > 0\,; \quad \frac{\partial T_2}{\partial t} = a_2 \frac{\partial^2 T_2}{\partial x^2}\,, \quad x > 0\,, \quad t > 0\,; \tag{1}$$

initial conditions

$$T_1 (x, 0) = T_{10} (x)\,, \quad x < 0\,; \quad T_2 (x, 0) = T_{20} (x)\,, \quad x > 0\,; \tag{2}$$

and by conjugate boundary conditions that represent heat transfer by the Newton law:

$$\lambda_1 \left.\frac{\partial T_1}{\partial x}\right|_{x=-0} = \lambda_2 \left.\frac{\partial T_2}{\partial x}\right|_{x=+0} = \alpha \left[T_2 (+0, t) - T_1 (-0, t)\right]. \tag{3}$$

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It is assumed that the temperatures in each half-space are limited in value:

$$T_1(x, t), \quad T_2(x, t) < \infty . \tag{4}$$

The existence and uniqueness theorem of solving heat conduction problems under the given conditions of conjugation and its proof are given, e.g., in [5], however, despite the simplicity of formulation of problem (1)–(3), its solution is absent at the present time. Conditions (3) are specific in that the temperature values on each side of the contact boundary are time-dependent. We will denote these, as yet unknown, conjugate functions as follows:

$$T_1(-0, t) = \varphi(t), \quad T_2(+0, t) = \psi(t) . \tag{5}$$

To find the solution of system (1)–(5) we will take the Laplace transform of (1)–(3) with respect to time that maps the time domain onto the region of the complex variable s:

$$T_L(x, s) = \int_0^\infty \exp(-st)\, T(x, t)\, dt .$$

Then, subject to (1)–(3), (5), we obtain

$$T''_{1L}(x, s) - \frac{s}{a_1} T_{1L}(x, s) + F_1(x) = 0 , \quad x < 0 , \quad T''_{2L}(x, s) - \frac{s}{a_2} T_{2L}(x, s) + F_2(x) = 0 , \quad x > 0 ; \tag{6}$$

$$T'_{1L}(-0, s) - H_1[\psi_L(s) - \varphi_L(s)] = 0 , \quad T'_{2L}(+0, s) + H_2[\varphi_L(s) - \psi_L(s)] = 0 ; \tag{7}$$

$$T_{1L}(-0, s) = \varphi_L(s), \quad T_{2L}(+0, s) = \psi_L(s) , \tag{8}$$

where $F_1(x) = T_{10}(x)/a_1$; $F_2(x) = T_{20}(x)/a_2$; $H_1 = \alpha/\lambda_1$; $H_2 = \alpha/\lambda_2$. Boundedness conditions (4) will go over into the following conditions:

$$T_1(x, s) , \quad T_2(x, s) < \infty . \tag{9}$$

By virtue of the linearity of boundary-value problems (6)–(7), we will represent their solution in the form

$$T_{1L}(x, s) = U_{1L}(x, s) + V_{1L}(x, s) , \quad x < 0 ; \quad T_{2L}(x, s) = U_{2L}(x, s) + V_{2L}(x, s) , \quad x > 0 , \tag{10}$$

where $U_{1L}(x, s)$ and $U_{2L}(x, s)$ are the solutions of the corresponding differential equations (6) under inhomogeneous boundary conditions (7) in the case where $F_1(x) = F_2(x) = 0$, i.e., the solutions of the problems (the fundamental solutions of the inhomogeneous boundary-value problem, e.g. [6])

$$U''_{1L} - k_1^2 U_{1L} = 0 , \quad x < 0 , \quad U'_{1L}(-0) + H_1 U_{1L}(-0) = H_1 \psi_L ; \tag{11}$$

$$U''_{2L} - k_2^2 U_{2L} = 0 , \quad x > 0 , \quad U'_{2L}(+0) - H_2 U_{2L}(+0) = H_2 \varphi_L , \tag{12}$$

where $k_1 = \sqrt{s/a_1}$ and $k_2 = \sqrt{s/a_2}$, whereas $U_{1L}(x, s)$ and $U_{2L}(x, s)$ are the solutions of inhomogeneous differential equations for transforms under inhomogeneous boundary conditions:

$$V''_{1L} - k_1^2 V_{1L} = -F_1(x) , \quad x < 0 , \quad V'_{1L}(-0) + H_1 V_{1L}(-0) = 0 ; \tag{13}$$

$$V''_{2L} - k_2^2 V_{2L} = -F_2(x) , \quad x > 0 , \quad V'_{2L}(+0) - H_2 V_{2L}(+0) = 0 . \tag{14}$$

We will represent the general solution of problem (11) in terms of the fundamental solution:

$$U_{1L} = \frac{H_1 \varphi_L}{k_1 + H_1} \exp(k_1 x) , \quad x < 0 . \tag{15}$$

Here, the boundedness condition (9) is used for $x \to -\infty$. Analogously, the general solution of problem (12) is written in the form

$$U_{2L} = -\frac{H_2 \varphi_L}{k_2 + H_2} \exp(-k_2 x) , \quad x > 0 . \tag{16}$$

Now, we will find the solution of problem (13), which is an inhomogeneous differential equation with a homogeneous boundary condition at the place of contact. We will represent the Green's function in terms of the fundamental solutions of the corresponding homogeneous equation in the following form:

$$G_1(x, \bar{x}) = \begin{cases} m_1(\bar{x}) \exp(k_1 x) + m_2(\bar{x}) \exp(-k_1 x) , & -\infty \le x \le \bar{x} \le -0 ; \\ n_1(\bar{x}) \exp(k_1 x) + n_2(\bar{x}) \exp(-k_1 x) , & -\infty \le \bar{x} \le x \le -0 . \end{cases} \tag{17}$$

On the basis of the general rule (e.g., [6]), the Green's function should be, first, continuous at the point $x = \bar{x}$, i.e.,

$$m_1 \exp(k_1 \bar{x}) + m_2 \exp(-k_1 \bar{x}) = n_1 \exp(k_1 \bar{x}) + n_2 \exp(-k_1 \bar{x}) ; \tag{18}$$

and, second, for a jump in the derivative G_1 over x at the point $x = \bar{x}$ the following equation should hold:

$$(n_1 - m_1) \exp(k_1 \bar{x}) - (n_2 - m_2) \exp(-k_1 \bar{x}) = \frac{1}{k_1} . \tag{19}$$

Moreover, the unknown Green's function must satisfy the boundary condition (the right-hand side expression of (13)) at the place of contact ($x = 0$), which, subject to the lower line in (17) and the expression for the derivative G'_{1x}, will yield

$$n_1 k_1 - n_2 k_1 + H_1 (n_1 + n_2) = 0 . \tag{20}$$

Taking into account the condition (14) of the boundedness of solution for $x \to -\infty$, from the upper line of (17) we may adopt at once that $G_1(-\infty, \bar{x}) = 0$, i.e.,

$$m_2 = 0\,. \tag{21}$$

The solution of system (18), (19) yields

$$m_2 - n_2 = \frac{1}{2k_1}\exp(k_1\bar{x})\,,\quad n_1 - m_1 = \frac{1}{2k_1}\exp(-k_1\bar{x})\,. \tag{22}$$

Thus, subject to (21), from (20) and (22) we find

$$n_2 = -\frac{1}{2k_1}\exp(k_1\bar{x})\,,\quad n_1 = \frac{H_1 - k_1}{2k_1(k_1 + H_1)}\exp(k_1\bar{x})\,,\quad m_1 = \frac{1}{2k_1}\left(\frac{H_1 - k_1}{k_1 + H_1}\exp(k_1\bar{x}) - \exp(-k_1\bar{x})\right). \tag{23}$$

Substituting (21) and (23) into (17), we write the Green's function in the form

$$G_1(x, \bar{x}) = \begin{cases} \dfrac{1}{2k_1}\left(\dfrac{k_1 - H_1}{k_1 + H_1}\exp(k_1\bar{x}) - \exp(-k_1\bar{x})\right)\exp(k_1 x)\,, & -\infty \le x \le \bar{x} \le -0\,; \\ \dfrac{1}{2k_1}\left(\dfrac{k_1 - H_1}{k_1 + H_1}\exp(k_1 x) - \exp(-k_1 x)\right)\exp(k_1\bar{x})\,, & -\infty \le \bar{x} \le x \le -0\,. \end{cases} \tag{24}$$

According to the well-known rule of constructing a solution of an inhomogeneous equation by a homogeneous one (problem (13)), using the Green's function (24) found, we write an equation for the transforms:

$$V_1(x) = -\int_{-\infty}^{-0} G(x, \bar{x})\,F_1(\bar{x})\,d\bar{x}\,. \tag{25}$$

Substituting (24) into (25), we obtain the solution of problem (13) in the region of transforms:

$$V_{1\mathrm{L}}(x, s) = \frac{1}{2k_1}\left[\exp(-k_1 x)\int_{-\infty}^{x}\exp(k_1\bar{x})\,F_1(\bar{x})\,d\bar{x} + \exp(k_1 x)\int_{x}^{-0}\exp(-k_1\bar{x})\,F_1(\bar{x})\,d\bar{x}\right.$$
$$\left. - \frac{H_1 - k_1}{H_1 + k_1}\exp(k_1 x)\int_{-\infty}^{x}\exp(k_1\bar{x})\,F_1(\bar{x})\,d\bar{x}\right],\quad x < 0\,. \tag{26}$$

The solution of problem (14) is written analogously:

$$V_{2\mathrm{L}}(x, s) = \frac{1}{2k_2}\left[\exp(k_2 x)\int_{x}^{+\infty}\exp(-k_2\bar{x})\,F_2(\bar{x})\,d\bar{x} + \exp(-k_2 x)\int_{+0}^{x}\exp(k_2\bar{x})\,F_2(\bar{x})\,d\bar{x}\right.$$
$$\left. - \frac{H_2 - k_2}{H_2 + k_2}\exp(-k_2 x)\int_{+0}^{+\infty}\exp(-k_2\bar{x})\,F_2(\bar{x})\,d\bar{x}\right],\quad x > 0\,. \tag{27}$$

According to (10), (15), and (26), the transform of the unknown solution of the basic problem in the negative half-space will have the form

$$T_{1\mathrm{L}}(x, s) = \frac{H_1\Psi_{\mathrm{L}}}{H_1 + k_1}\exp(k_1 x) + \frac{1}{2k_1}\left[\frac{k_1 - H_1}{H_1 + k_1}\exp(k_1 x)\int_{-\infty}^{0}\exp(k_1\bar{x})\,F_1(\bar{x})\,d\bar{x}\right.$$

$$+ \exp(-k_1 x) \int_{-\infty}^{x} \exp(k_1 \bar{x}) F_1(\bar{x}) d\bar{x} + \exp(k_1 x) \int_{x}^{-0} \exp(-k_1 \bar{x}) F_1(\bar{x}) d\bar{x} \Bigg] , \quad x < 0 , \tag{28}$$

whereas, according to (10), (16), and (27), the transform in the positive half-space has the form

$$T_{2L}(x, s) = \frac{H_2 \varphi_L}{H_2 + k_2} \exp(-k_2 x) + \frac{1}{2k_2} \Bigg[\frac{k_2 - H_2}{H_2 + k_2} \exp(-k_2 x) \int_{0}^{+\infty} \exp(-k_2 \bar{x}) F_2(\bar{x}) d\bar{x}$$

$$+ \exp(k_2 x) \int_{x}^{+\infty} \exp(-k_2 \bar{x}) F_2(\bar{x}) d\bar{x} + \exp(-k_2 x) \int_{+0}^{x} \exp(k_2 \bar{x}) F_2(\bar{x}) d\bar{x} \Bigg] , \quad x > 0 . \tag{29}$$

Based on the notation (5) introduced, as well as on the unknown functions (28) and (29) obtained, we have

$$T_{1L} \big|_{x=-0} = \varphi_L = \frac{1}{H_1 + k_1} \left[H_1 \psi_L + \int_{-\infty}^{-0} \exp(k_1 \bar{x}) F_1(\bar{x}) d\bar{x} \right] , \tag{30}$$

$$T_{2L} \big|_{x=+0} = \psi_L = \frac{1}{H_2 + k_2} \left[H_2 \varphi_L + \int_{+0}^{+\infty} \exp(-k_2 \bar{x}) F_2(\bar{x}) d\bar{x} \right] . \tag{31}$$

Thus, for determining the functions φ_L and ψ_L we have a system of two equations (30) and (31). If in these equations we go over from the functions $F_1(\bar{x})$ and $F_2(\bar{x})$ to $T_1(\bar{x})$ and $T_2(\bar{x})$ and replace k_1 and k_2 (except for the terms with exponents) by their values given in terms of s, a_1, and s_2, then this system acquires the form

$$\left(1 + \frac{H_1}{\sqrt{s/a_1}} \right) \varphi_L - \frac{H_1}{\sqrt{s/a_1}} \psi_L = I_{10}(s) , \quad - \frac{H_2}{\sqrt{s/a_2}} \varphi_L + \left(1 + \frac{H_2}{\sqrt{s/a_2}} \right) \psi_L = I_{20}(s) , \tag{32}$$

where

$$I_{10}(s) = \frac{1}{\sqrt{a_1 s}} \int_{-\infty}^{-0} \exp(k_1 \bar{x}) T_1(\bar{x}) d\bar{x} , \quad I_{20}(s) = \frac{1}{\sqrt{a_2 s}} \int_{+0}^{+\infty} \exp(-k_2 \bar{x}) T_2(\bar{x}) d\bar{x} . \tag{33}$$

The solutions of system (32) for the transforms of unknown functions are given by the equations

$$\varphi_L = \frac{I_{10}(\sqrt{s} + H_2 \sqrt{a_2}) + I_{10} H_2 \sqrt{a_2}}{\sqrt{s} + H_2 \sqrt{a_2} + H_1 \sqrt{a_1}} , \quad \psi_L = \frac{I_{20}(\sqrt{s} + H_1 \sqrt{a_1}) + I_{10} H_2 \sqrt{a_2}}{\sqrt{s} + H_2 \sqrt{a_2} + H_1 \sqrt{a_1}} . \tag{34}$$

Substituting (34) into (28) and (29), taking into account the notation adopted, we obtain the transforms of the unknown temperature distributions:

$$T_{1L}(x, s) = \frac{\exp(k_1 x)}{H_1 k_2 + H_2 k_1 + k_1 k_2} \left[\frac{H_2 k_1 + k_2 (k_1 - H_1)}{2k_1} i_1 + H_1 i_2 \right]$$

$$+ \frac{1}{2k_1} (\exp(-k_1 x) \overset{x}{i_1} + \exp(k_1 x) i_{1x}) , \quad x < 0 ; \tag{35}$$

$$T_{2L}(x, s) = \frac{\exp(-k_2 x)}{H_1 k_2 + H_2 k_1 + k_1 k_2} \left[\frac{H_1 k_2 + k_1 (k_2 - H_2)}{2k_2} i_2 + H_2 i_1 \right]$$

$$+\frac{1}{2k_2}(\exp(k_2x)\, i_{2x}+\exp(-k_2x)\, i_2^x)\,,\quad x>0\,, \tag{36}$$

where the following notations are introduced:

$$i_1=\int_{-\infty}^{-0}\exp(k_1\bar{x})\, F_1(\bar{x})\, d\bar{x}\,,\quad i_2=\int_{+0}^{+\infty}\exp(-k_2\bar{x})\, F_2(\bar{x})\, d\bar{x}\,,\quad i_1^x=\int_{-\infty}^{x}\exp(k_1\bar{x})\, F_1(\bar{x})\, d\bar{x}\,,$$

$$i_{1x}=\int_{x}^{-0}\exp(-k_1\bar{x})\, F_1(\bar{x})\, d\bar{x}\,,\quad i_2^x=\int_{+0}^{x}\exp(k_2\bar{x})\, F_2(\bar{x})\, d\bar{x}\,,\quad i_{2x}=\int_{x}^{+\infty}\exp(-k_2\bar{x})\, F_2(\bar{x})\, d\bar{x}\,, \tag{37}$$

with $I_{10} = i_1\sqrt{a_1/s}$ and $I_{20} = i_2\sqrt{a_2/s}$. The symbol x at i_1 and i_2 is an index and not an exponent. We will write the final expression for T_{1L} and T_{2L}, having replaced k_1 and k_2 in (35) and (36) by their values $k_1(s)$ and $k_2(s)$:

$$T_{1L}(x,s)=\frac{\sqrt{a_1}}{2}\left[\frac{H_2\sqrt{a_2}-H_1\sqrt{a_1}+\sqrt{s}}{\sqrt{s}\,(H_1\sqrt{a_1}+H_2\sqrt{a_2}+\sqrt{s})}\exp(\sqrt{s/a_1}\,x)\, i_1\right.$$

$$\left.+\frac{2H_1\sqrt{a_2}}{\sqrt{s}\,(H_1\sqrt{a_1}+H_2\sqrt{a_2}+\sqrt{s})}\exp(\sqrt{s/a_1}\,x)\, i_2+\frac{1}{\sqrt{s}}\left(\exp(-\sqrt{s/a_1}\,x)\, i_1^x+\exp(\sqrt{s/a_1}\,x)\, i_{1x}\right)\right],\quad x<0\,. \tag{38}$$

In what follows, equations for temperatures and their transforms are given only for the negative half-space, since for the positive one they are symmetric about indices. Substituting (37) into (38), we find

$$T_{1L}(x,s)=\frac{\sqrt{a_1}}{2}\left[\int_{-\infty}^{-0}\frac{H_2\sqrt{a_2}-H_1\sqrt{a_1}+\sqrt{s}}{\sqrt{s}\,(b+\sqrt{s})}\exp(\sqrt{s/a_1}\,(x+\bar{x}))\, F_1(\bar{x})\, d\bar{x}\right.$$

$$+2H_1\sqrt{a_2}\int_{+0}^{+\infty}\frac{1}{\sqrt{s}\,(b+\sqrt{s})}\exp\left(\left(\frac{x}{\sqrt{a_1}}-\frac{\bar{x}}{\sqrt{a_2}}\right)\sqrt{s}\right)F_2(\bar{x})\, d\bar{x}$$

$$\left.+\int_{-\infty}^{x}\frac{1}{\sqrt{s}}\exp\left(\left(\frac{x}{\sqrt{a_1}}-\frac{\bar{x}}{\sqrt{a_2}}\right)\sqrt{s}\right)F_1(\bar{x})\, d\bar{x}+\int_{x}^{-0}\frac{1}{\sqrt{s}}\exp(-\sqrt{s/a_1}\,(\bar{x}-x))\, F_1(\bar{x})\, d\bar{x}\right],\quad x<0\,. \tag{39}$$

Passing in expression (39) from the functions $F_1(\bar{x})$ and $F_2(\bar{x})$ to $T_{10}(\bar{x})$ and $T_{20}(\bar{x})$ and calculating the inverse transform, we obtain the sought-for distribution of temperature in the negative half-space:

$$T_1(x,t)=H_1\left\{-\int_{-\infty}^{-0}\exp\left(b^2t-b\frac{x+\bar{x}}{\sqrt{a_1}}\right)\operatorname{erfc}\left(b\sqrt{t}-\frac{x+\bar{x}}{2\sqrt{a_1t}}\right)T_{10}(\bar{x})\, d\bar{x}\right.$$

$$\left.+\sqrt{\frac{a_1}{a_2}}\int_{+0}^{+\infty}\exp\left(b^2t-b\left(\frac{x}{\sqrt{a_1}}-\frac{\bar{x}}{\sqrt{a_2}}\right)\right)\operatorname{erfc}\left[b\sqrt{t}-\left(\frac{x}{\sqrt{a_1}}-\frac{\bar{x}}{\sqrt{a_2}}\right)\frac{1}{2\sqrt{t}}\right]T_{20}(\bar{x})\, d\bar{x}\right\}$$

$$+\frac{1}{2\sqrt{\pi a_1t}}\int_{-\infty}^{-0}\left[\exp\left(-\frac{(x+\bar{x})^2}{4a_1t}\right)+\exp\left(-\frac{(x-\bar{x})^2}{4a_1t}\right)\right]T_{10}(\bar{x})\, d\bar{x}\,,\quad x<0\,. \tag{40}$$

Based on the above-mentioned principle of symmetry, the sought-for temperature distribution for the positive half-space is written in the form

$$T_2(x,t) = H_2 \left\{ -\int_{+0}^{+\infty} \exp\left(b^2 t + b\,\frac{x+\bar{x}}{\sqrt{a_1}}\right) \operatorname{erfc}\left(b\sqrt{t} + \frac{x+\bar{x}}{2\sqrt{a_1 t}}\right) T_{20}(\bar{x})\, d\bar{x}\right.$$

$$\left. + \sqrt{\frac{a_2}{a_1}} \int_{-\infty}^{-0} \exp\left(b^2 t - b\left(\frac{\bar{x}}{\sqrt{a_1}} - \frac{x}{\sqrt{a_2}}\right)\right) \operatorname{erfc}\left[b\sqrt{t} - \left(\frac{\bar{x}}{\sqrt{a_1}} - \frac{x}{\sqrt{a_2}}\right)\frac{1}{2\sqrt{t}}\right] T_{10}(\bar{x})\, d\bar{x}\right\}$$

$$+ \frac{1}{2\sqrt{\pi a_2 t}} \int_{+0}^{+\infty} \left[\exp\left(-\frac{(\bar{x}+x)^2}{4a_2 t}\right) + \exp\left(-\frac{(\bar{x}-x)^2}{4a_2 t}\right)\right] T_{20}(\bar{x})\, d\bar{x}\,, \quad x > 0\,. \tag{41}$$

We can check the validity of the solutions obtained by calculating the above-given integrals at constant initial temperatures ($T_{10} = T_{10}^* = \text{const}$ and $T_{20} = T_{20}^* = \text{const}$). In this case, Eqs. (40) and (41) yield an accurate solution [7].

A Particular Case of Exponential Initial Conditions. We will consider the exponential distribution of initial temperatures:

$$T_1(x,0) = T_{10}(x) = T_{10}^* \exp(h_1 x)\,, \quad x < 0\,, \quad h_1 > 0\,;$$

$$T_2(x,0) = T_{20}(x) = T_{20}^* \exp(-h_2 x)\,, \quad x > 0\,, \quad h_2 > 0\,. \tag{42}$$

By virtue of the symmetry of physical processes about the plane $x = 0$, it is sufficient to restrict ourselves to searching for a solution only for the negative half-space. We write it in the form

$$T_1(x,t) = H_1\left(J_1 + \sqrt{\frac{a_1}{a_2}}\, J_2\right) + \frac{1}{2\sqrt{\pi a_1 t}}\, J_3\,, \tag{43}$$

where J_1, J_2, and J_3 are the first, second, and third integral expressions in solution (40). To calculate the first integral, we introduce the change of variables and transformations:

$$b\sqrt{t} - \frac{x+\bar{x}}{2\sqrt{a_1 t}} = y\,, \quad \bar{x} = 2b\sqrt{a_1 t} - x - 2\sqrt{a_1 t}\, y\,, \quad d\bar{x} = -2\sqrt{a_1 t}\, dy\,, \quad y\big|_{\bar{x}=-\infty} = +\infty\,, \quad y\big|_{\bar{x}=-0} = b\sqrt{t} - \frac{x}{2\sqrt{a_1 t}}\,.$$

Then

$$J_1 = -\int_{-\infty}^{-0} \exp\left(b^2 t - b\,\frac{x+\bar{x}}{\sqrt{a_1}}\right) \operatorname{erfc}\left(b\sqrt{t} - \frac{x+\bar{x}}{2\sqrt{a_1}}\right) T_{10}^* \exp(h_1 \bar{x})\, d\bar{x}$$

$$= -2T_{10}^* \sqrt{a_1 t}\, \exp\left(-b^2 t + h_1(2b\sqrt{a_1 t} - x)\right) \int_{b\sqrt{t} - \frac{x}{2\sqrt{a_1 t}}}^{+\infty} \exp\left(2(b - h_1\sqrt{a_1})\sqrt{t}\, y\right) \operatorname{erfc}(y)\,.$$

In the second integral J_2 we introduce the change of variables and transformations:

$$b\sqrt{t}-\left(\frac{x}{\sqrt{a_1}}-\frac{\bar{x}}{\sqrt{a_2}}\right)\frac{1}{2\sqrt{t}}=z\,,\quad \bar{x}=2b\sqrt{a_2t}+\sqrt{\frac{a_2}{a_1}}\,x+2\sqrt{a_2t}\,z\,,$$

$$d\bar{x}=2\sqrt{a_2t}\,dz\,,\quad z\big|_{\bar{x}=+0}=b\sqrt{t}-\frac{x}{2\sqrt{a_1t}}\,,\quad z\big|_{\bar{x}=+\infty}=+\infty\,.$$

In such a case

$$J_2=\int_{+0}^{+\infty}\exp\left(b^2t-b\left(\frac{x}{\sqrt{a_1}}-\frac{\bar{x}}{\sqrt{a_2}}\right)\right)\operatorname{erfc}\left[b\sqrt{t}-\left(\frac{x}{\sqrt{a_1}}-\frac{\bar{x}}{\sqrt{a_2}}\right)\frac{1}{2\sqrt{t}}\right]T^*_{20}\exp\left(-h_2\bar{x}\right)d\bar{x}$$

$$=2T^*_{20}\sqrt{a_2t}\exp\left(-b^2t+h_2\left(2b\sqrt{a_2t}-\sqrt{\frac{a_2}{a_1}}\,x\right)\right)\int_{b\sqrt{t}-\frac{x}{2\sqrt{a_1t}}}^{+\infty}\exp\left(2\left(b-h_2\sqrt{a_2}\right)\sqrt{t}\,z\right)\operatorname{erfc}(z)\,dz\,.$$

Now, we determine the expressions in the parentheses in Eq. (43):

$$J_1+\sqrt{\frac{a_2}{a_1}}\,J_2=2\sqrt{a_1t}\exp\left(-b^2t\right)\left[T^*_{20}\exp\left(h_2\left(2b\sqrt{a_2t}-\sqrt{\frac{a_2}{a_1}}\,x\right)\right)\int_{b\sqrt{t}-\frac{x}{2\sqrt{a_1t}}}^{+\infty}\exp\left(2\left(b-h_2\sqrt{a_2}\right)\sqrt{t}\,z\right)\operatorname{erfc}(z)\,dz\right.$$

$$\left.-T^*_{10}\exp\left(h_1\left(2b\sqrt{a_1t}-x\right)\right)\int_{b\sqrt{t}-\frac{x}{2\sqrt{a_1t}}}^{+\infty}\exp\left(2\left(b-h_1\sqrt{a_1}\right)\sqrt{t}\,y\right)\operatorname{erfc}(y)\,dy\right].\qquad(44)$$

We calculate "by parts" the first integral in equality (44), having denoted it by J_4:

$$J_4=\int_{b\sqrt{t}-\frac{x}{2\sqrt{a_1t}}}^{+\infty}\exp\left(2\left(b-h_2\sqrt{a_2}\right)\sqrt{t}\,z\right)\operatorname{erfc}(z)\,dz$$

$$=\frac{1}{2\left(b-h_2\sqrt{a_2}\right)\sqrt{t}}\left[\operatorname{erfc}(z)\exp\left(2\left(b-h_2\sqrt{a_2}\right)\sqrt{t}z\right)\Big|_{b\sqrt{t}-\frac{x}{2\sqrt{a_1t}}}^{+\infty}\right.$$

$$\left.-\int_{b\sqrt{t}-\frac{x}{2\sqrt{a_1t}}}^{+\infty}\exp\left(2\left(b-h_2\sqrt{a_2}\right)\sqrt{t}\,z\right)\operatorname{erfc}(z)\,dz\right].$$

Using the expression of the function erfc (z) at high values of z (which has already been used earlier), it can be easily shown that

$$\operatorname{erfc}(z)\exp\left(2\,(b-h_2\sqrt{a_2})\sqrt{t}\,z\right)\Big|_{z=+\infty}=0\,.$$

Taking into account well-known rule

$$\frac{d}{dz}\operatorname{erfc}(z)=-\operatorname{erf}(z)=-\frac{2}{\sqrt{\pi}}\exp(-z^2)\,,$$

we obtain

$$J_4=\frac{1}{2\,(b-h_2\sqrt{a_2})\sqrt{t}}\left[-\exp\left(2\,(b-h_2\sqrt{a_2})\sqrt{t}\left(b\sqrt{t}-\frac{x}{2\sqrt{a_1 t}}\right)\right)\operatorname{erfc}\left(b\sqrt{t}-\frac{x}{2\sqrt{a_1 t}}\right)\right.$$

$$\left.+\frac{2}{\pi}\int\limits_{b\sqrt{t}-\frac{x}{2\sqrt{a_1 t}}}^{+\infty}\exp\left(2\,(b-h_2\sqrt{a_2})\sqrt{t}\,z-z^2\right)dz\right].$$

According to the reference data [8], the integral in the last expression is equal to

$$\int\limits_{b\sqrt{t}-\frac{x}{2\sqrt{a_1 t}}}^{+\infty}\exp\left(2\,(b-h_2\sqrt{a_2})\sqrt{t}\,z-z^2\right)dz=\frac{\sqrt{\pi}}{2}\exp\left(2\,(b-h_2\sqrt{a_2}\,)^2\,t\right)\operatorname{erfc}\left(h_2\sqrt{a_2 t}-\frac{x}{2\sqrt{a_1 t}}\right),$$

therefore

$$J_4=\frac{1}{2\,(b-h_2\sqrt{a_2})\sqrt{t}}\left[-\exp\left(2\,(b-h_2\sqrt{a_2})\sqrt{t}\left(b\sqrt{t}-\frac{x}{2\sqrt{a_1 t}}\right)\right)\operatorname{erfc}\left(b\sqrt{t}-\frac{x}{2\sqrt{a_1 t}}\right)\right.$$

$$\left.+\exp\left(2\,(b-h_2\sqrt{a_2})^2\,t\right)\operatorname{erfc}\left(h_2\sqrt{a_2 t}-\frac{x}{2\sqrt{a_1 t}}\right)\right].\qquad(45)$$

Along with the integral J_4 we will consider the second integral J_4^* which also enters into equality (44):

$$J_4^*=\int\limits_{b\sqrt{t}-\frac{x}{2\sqrt{a_1 t}}}^{+\infty}\exp\left(2\,(b-h_1\sqrt{a_1})\sqrt{t}\,y\right)\operatorname{erfc}(y)\,dy\,.$$

Just as in the calculation of J_4, we will apply the rule of integration by parts; then, based on the reference data [8] and the above-indicated properties of the function erfc (z), we obtain

$$J_4^*=\frac{1}{2\,(b-h_1\sqrt{a_1})\sqrt{t}}\left[-\exp\left(2\,(b-h_1\sqrt{a_1})\sqrt{t}\left(b\sqrt{t}-\frac{x}{2\sqrt{a_1 t}}\right)\right)\operatorname{erfc}\left(b\sqrt{t}-\frac{x}{2\sqrt{a_1 t}}\right)\right.$$

$$\left.+\exp\left(2\,(b-h_1\sqrt{a_1})^2\,t\right)\operatorname{erfc}\left(h_1\sqrt{a_1 t}-\frac{x}{2\sqrt{a_1 t}}\right)\right].\qquad(46)$$

Using (45) and (46), expression (44) can be rearranged as

$$J_1 + \sqrt{\frac{a_2}{a_1}}\, J_2 = \sqrt{a_1}\left[\left(\frac{T_{10}^*}{b - h_1\sqrt{a_1}} - \frac{T_{20}^*}{b - h_2\sqrt{a_2}}\right)\exp\left(b^2 t - \frac{bx}{\sqrt{a_1}}\right)\operatorname{erfc}\left(b\sqrt{t} - \frac{x}{2\sqrt{a_1 t}}\right)\right.$$

$$\left. + \frac{T_{10}^*}{b - h_1\sqrt{a_1}}\exp\left(h_1^2 a_1 t - h_1 x\right)\operatorname{erfc}\left(h_1\sqrt{a_1 t} - \frac{x}{2\sqrt{a_1 t}}\right)\right]. \quad (47)$$

Now, we calculate the last, third, integral in expression (43):

$$J_3 = \int_{-\infty}^{-0}\left[\exp\left(-\frac{(x+\bar{x})^2}{4a_1 t}\right) + \exp\left(-\frac{(x-\bar{x})^2}{4a_1 t}\right)\right]\exp\left(h_1\bar{x}\right) d\bar{x}\,. \quad (48)$$

First, we consider the integral of the first term in (48) (having denoted it by J_3^*). As before, we introduce the change of variable and the transformations

$$\frac{x+\bar{x}}{2\sqrt{a_1 t}} = y\,,\quad d\bar{x} = 2\sqrt{a_1 t}\, dy\,,\quad \bar{x} = -x - 2\sqrt{a_1 t}\, y\,,\quad y\big|_{\bar{x}=-\infty} = -\infty\,,\quad y\big|_{\bar{x}=-0} = \frac{x}{2\sqrt{a_1 t}}\,.$$

Then

$$J_3^+ = \int_{-\infty}^{-0}\exp\left(-\frac{(x+\bar{x})^2}{4a_1 t}\right)\exp\left(h_1\bar{x}\right) d\bar{x} = 2\sqrt{a_1 t}\int_{-\frac{x}{2\sqrt{a_1 t}}}^{+\infty}\exp\left(-y^2 + 2h_1\sqrt{a_1 t}\, y\right) dy\,.$$

Using the tabular data of [8], for this expression we obtain

$$J_3^+ = \sqrt{\pi a_1 t}\,\exp\left(h_1\left(h_1 a_1 t - x\right)\right)\operatorname{erfc}\left(h_1\sqrt{a_1 t} - \frac{x}{2\sqrt{a_1 t}}\right). \quad (49)$$

The integral involving the second term in (48) (denoted by J_3^-) is calculated similarly to J_3^+. We introduce the change of variables and transformation:

$$\frac{x+\bar{x}}{2\sqrt{a_1 t}} = z\,,\quad d\bar{x} = -2\sqrt{a_1 t}\, dz\,,\quad \bar{x} = x - 2\sqrt{a_1 t}\, z\,,\quad z\big|_{\bar{x}=-\infty} = -\infty\,,\quad z\big|_{\bar{x}=-0} = \frac{x}{2\sqrt{a_1 t}}\,.$$

As a result

$$J_3^- = \int_{-\infty}^{-0}\exp\left(-\frac{(x-\bar{x})^2}{4a_1 t}\right)\exp\left(h_1\bar{x}\right) d\bar{x} = \sqrt{\pi a_1 t}\,\exp\left(h_1\left(h_1 a_1 t + x\right)\right)\operatorname{erfc}\left(h_1\sqrt{a_1 t} + \frac{x}{2\sqrt{a_1 t}}\right). \quad (50)$$

Summing up (49) and (50), we obtain

$$J_3 = J_3^+ + J_3^- = \sqrt{\pi a_1 t}\,\exp\left(h_1^2 a_1 t\right)$$

$$\times\left[\exp\left(h_1 x\right)\operatorname{erfc}\left(h_1\sqrt{a_1 t} + \frac{x}{2\sqrt{a_1 t}}\right) + \exp\left(-h_1 x\right)\operatorname{erfc}\left(h_1\sqrt{a_1 t} - \frac{x}{2\sqrt{a_1 t}}\right)\right]. \quad (51)$$

Subject to (47) and (51), the sought-for solution (43) is transformed as

$$T_1(x,t) = H_1\sqrt{a_1}\left[\left(\frac{T_{10}^*}{b-h_1\sqrt{a_1}} - \frac{T_{20}^*}{b-h_2\sqrt{a_2}}\right)\exp\left(b^2 t - \frac{bx}{\sqrt{a_1}}\right)\operatorname{erfc}\left(b\sqrt{t} - \frac{x}{2\sqrt{a_1 t}}\right)\right.$$

$$+\frac{T_{20}^*}{b-h_2\sqrt{a_2}}\exp\left(h_2^2 a_2 t - h_2\sqrt{a_2/a_1}\,x\right)\operatorname{erfc}\left(h_2\sqrt{a_2 t} - \frac{x}{2\sqrt{a_1 t}}\right)$$

$$\left.-\frac{T_{10}^*}{b-h_1\sqrt{a_1}}\exp\left(h_1^2 a_1 t - h_1 x\right)\operatorname{erfc}\left(h_1\sqrt{a_1 t} - \frac{x}{2\sqrt{a_1 t}}\right)\right]$$

$$+\frac{T_{10}^*}{2}\exp(h_1^2 a_1 t)\left[\exp(h_1 x)\operatorname{erf}\left(h_1\sqrt{a_1 t} + \frac{x}{2\sqrt{a_1 t}}\right) + \exp(-h_1 x)\operatorname{erf}\left(h_1\sqrt{a_1 t} - \frac{x}{2\sqrt{a_1 t}}\right)\right], \quad x < 0\,. \tag{52}$$

By virtue of the symmetry of the problem about the plane $x = 0$, the solution for the positive half-space can be written on the basis of (52) as

$$T_2(x,t) = H_2\sqrt{a_2}\left[\left(\frac{T_{20}^*}{b-h_2\sqrt{a_2}} - \frac{T_{10}^*}{b-h_1\sqrt{a_1}}\right)\exp\left(b^2 t + \frac{bx}{\sqrt{a_2}}\right)\operatorname{erfc}\left(b\sqrt{t} + \frac{x}{2\sqrt{a_2 t}}\right)\right.$$

$$+\frac{T_{10}^*}{b-h_1\sqrt{a_1}}\exp\left(h_1^2 a_1 t + h_1\sqrt{a_1/a_2}\,x\right)\operatorname{erfc}\left(h_1\sqrt{a_1 t} + \frac{x}{2\sqrt{a_2 t}}\right)$$

$$\left.-\frac{T_{20}^*}{b-h_2\sqrt{a_2}}\exp\left(h_2^2 a_2 t + h_1 x\right)\operatorname{erfc}\left(h_2\sqrt{a_2 t} + \frac{x}{2\sqrt{a_2 t}}\right)\right]$$

$$+\frac{T_{20}^*}{2}\exp(h_2^2 a_2 t)\left[\exp(-h_2 x)\operatorname{erf}\left(h_2\sqrt{a_2 t} - \frac{x}{2\sqrt{a_2 t}}\right) + \exp(h_2 x)\operatorname{erf}\left(h_2\sqrt{a_2 t} + \frac{x}{2\sqrt{a_2 t}}\right)\right], \quad x > 0\,. \tag{53}$$

Checking the Solution (the Case of Constant Initial Temperatures). The correctness of the solution obtained can be verified by considering the case of constant initial temperatures, i.e., where $h_1 = h_2 = 0$. Substitution of these values into (52) and (53) yields

$$T_1(x,t) = T_{10}^* - \frac{T_{10}^* - T_{20}^*}{1+K_{\varepsilon_1}}\left[\operatorname{erfc}\left(-\frac{x}{2\sqrt{a_1 t}}\right) - \exp\left(b^2 t - \frac{bx}{\sqrt{a_1}}\right)\operatorname{erfc}\left(b\sqrt{t} - \frac{x}{2\sqrt{a_1 t}}\right)\right], \quad x < 0\,; \tag{54}$$

$$T_2(x,t) = T_{20}^* - \frac{T_{20}^* - T_{10}^*}{1+K_{\varepsilon_2}}\left[\operatorname{erfc}\left(-\frac{x}{2\sqrt{a_2 t}}\right) - \exp\left(b^2 t + \frac{bx}{\sqrt{a_2}}\right)\operatorname{erfc}\left(b\sqrt{t} + \frac{x}{2\sqrt{a_2 t}}\right)\right], \quad x > 0\,, \tag{55}$$

where the following connections between the parameters were used:

$$\frac{H_1\sqrt{a_1}}{b} = \frac{1}{1+K_{\varepsilon_1}}\,; \quad \frac{H_2\sqrt{a_2}}{b} = \frac{1}{1+K_{\varepsilon_2}}\,.$$

Conclusions. Thus, expressions (52) and (53) go over into the accurate solution (54), (55) obtained for this case earlier [7], pointing to the validity of the analytical solution (52), (53). Equations (54) and (55) can also been obtained directly from (40), (41), which proves the validity of the integral representation of (40), (41).

NOTATION

$a = \lambda/(c\gamma)$, thermal diffusivity; b, thermal mutual influence of half-spaces which depends on the degree of the nonideality of their contact; c, isochoric specific heat; $F_1(x) = T_{10}(x)/a_1$; $F_2(x) = T_{20}(x)/a_2$; $H = \alpha/\lambda$, reduced (relative) heat transfer coefficient; K_{ε_1} and K_{ε_2}, criteria of the thermal activity of the first half-space relative to the second and the second relative to the first one, respectively; s, complex variable in the Laplace transform; T, temperature; t, time; x, coordinate; α, heat transfer coefficient; γ, density; ε, thermal activity; λ, thermal conductivity; φ and ψ, conjugate functions equal to the values of T_1 and T_2 at $x = 0$, respectively. Subscripts: 1 and 2, the first and second half-spaces, respectively; 10 and 20, initial values related to the first and second half-spaces, respectively; L, sign of Laplace transform.

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